Tutorial Notes 6

1. Find the volume of the region: $0 \le z \le x^2$, $x \le y \le 6 - x^2$.

   Solutions:

   Note that the range of x is $x \le 6 - x^2$: $-3 \le x \le 2$. The volume is
   $$\int_{-3}^{2}\int_{x}^{6-x^2}\int_{0}^{x^2} \mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x = \int_{-3}^{2} x^2(6 - x^2 - x)\,\mathrm{d}x = \frac{125}{4}.$$

2. Evaluate
   $$\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \log(x^2 + y^2 + 1)\,\mathrm{d}x\,\mathrm{d}y.$$

   Solutions:

   The region is $x^2 + y^2 \le 1$. Using polar coordinates, the integral is
   $$\begin{aligned}
   &\int_{0}^{1}\int_{0}^{2\pi} \log(r^2 + 1)\cdot r\,\mathrm{d}\theta\,\mathrm{d}r \\
   &= \pi\int_{0}^{1} \log(r^2 + 1)\,\mathrm{d}(r^2) \\
   &= \pi\int_{0}^{1} \log(u + 1)\,\mathrm{d}u \\
   &= \pi\left(u\log(u + 1)|_0^1 - \int_0^1 \frac{u}{u+1}\,\mathrm{d}u\right) \\
   &= \pi(\log 2 - 1 + \log 2) \\
   &= \pi(2\log 2 - 1).
   \end{aligned}$$

3. Find the average of $f(x, y, z) = 30xz\sqrt{x^2 + y}$ on the region: $0 \le x \le 1$, $0 \le y \le 3$, $0 \le z \le 1$.

   Solutions:

   The volume is 3. It suffices to calculate
   $$\begin{aligned}
   &\int_{0}^{3}\int_{0}^{1}\int_{0}^{1} 30xz\sqrt{x^2 + y}\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y \\
   &= 15\int_{0}^{3}\int_{0}^{1} x\sqrt{x^2 + y}\,\mathrm{d}x\,\mathrm{d}y \\
   &= 15\int_{0}^{3}\int_{0}^{1} \sqrt{x^2 + y}\,\mathrm{d}(x^2/2)\,\mathrm{d}y
   \end{aligned}$$

$$\begin{aligned}
&= \frac{15}{2}\int_0^3\int_0^1 \sqrt{u+y}\,\mathrm{d}u\,\mathrm{d}y \\
&= \frac{15}{2}\int_0^3 \frac{2}{3}[(1+y)^{3/2} - y^{3/2}]\,\mathrm{d}y \\
&= 5\cdot\frac{2}{5}[4^{5/2} - 1 - 3^{5/2}] \\
&= 2(31 - 9\sqrt{3}).
\end{aligned}$$

Hence the average value is $2(31 - 9\sqrt{3})/3$.

4. Transfer the integral

$$\int_0^{\pi/2}\int_1^{\sqrt{3}}\int_1^{\sqrt{4-r^2}} r^3(\sin\theta\cos\theta)z^2\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta.$$

given in cylindrical coordinates to rectangular coordinates. The order in rectangular coordinates is z first, then y, then x.

Solutions:

The region is $0 \le \theta \le \pi/2$, $1 \le r \le \sqrt{3}$, $1 \le z \le \sqrt{4-r^2}$, which is equivalent to x, $y \ge 0$, $1 \le x^2+y^2 \le 3$, $1 \le z \le \sqrt{4-x^2-y^2}$. Then the region is equivalent to

$$\begin{aligned}
1 \le z \le \sqrt{4-x^2-y^2}&, \\
\sqrt{1-x^2} \le y \le \sqrt{3-x^2}&, \quad \text{if } 0 \le x \le 1, \\
0 \le y \le \sqrt{3-x^2}&, \quad \text{if } 1 \le x \le \sqrt{3}.
\end{aligned}$$

Hence the integral in rectangular coordinates is

$$\int_0^1\int_{\sqrt{1-x^2}}^{\sqrt{3-x^2}}\int_1^{\sqrt{4-x^2-y^2}} xyz^2\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x + \int_1^{\sqrt{3}}\int_0^{\sqrt{3-x^2}}\int_1^{\sqrt{4-x^2-y^2}} xyz^2\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x.$$

Moreover, the value of the integral is $(39\sqrt{3} - 31)/70$.

5. Prove that for a continuous function f,

$$\int_0^\infty\int_0^x \mathrm{e}^{-sx} f(x-y,y)\,\mathrm{d}y\,\mathrm{d}x = \int_0^\infty\int_0^\infty \mathrm{e}^{-s(x+y)} f(x,y)\,\mathrm{d}x\,\mathrm{d}y.$$

Solutions:

Let $u = x-y$ and $v = y$. Then $x = u+v$, $y = v$ and the Jacobian $|\partial(u,v)/\partial(x,y)| = 1$. In the uv coordinate, the region is $u \ge 0$, $v \ge 0$. Hence by the change of variable formula, the integral is

$$\int_0^\infty\int_0^\infty \mathrm{e}^{-s(u+v)} f(u,v)\,\mathrm{d}u\,\mathrm{d}v.$$